The Automorphic Transformations of the Binary Quartic.

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The transformations which are the subject of this paper have been discussed from several points of view. They form groups which are holohedrically isomorphic with certain substitution groups of four letters; and they have been given a picturesque geometric interpretation by use of the regular polyhedrons. When, however, the analytical expressions of these transformations have been obtained, it has always been by a specialization of the system of coordinates, which reduces the quartic to a canonical form. It is the object here to derive them for the general quartic.

1. *Substitution groups.* If, as usual, we let $[ABCD]$ denote a definite double ratio of four distinct points on a line, then the following permutations will in every case have the same value:

$$[ABCD], \qquad [BADC], \qquad [CDAB], \qquad [DCBA].$$

If the four points have the harmonic position, so that $[ABCD]=-1$, there are eight permutations which have the same value:

$$\begin{array}{cccc} [ABCD], & [BADC], & [CDAB], & [DCBA], \\ [BACD], & [ABDC], & [CDBA], & [DCAB]. \end{array}$$

For the equianharmonic position the letters may be permuted in twelve ways without changing the value of the double ratio:

$$\begin{array}{cccc} [ABCD], & [BADC], & [CDAB], & [DCBA], \\ [BCAD], & [CBDA], & [ADBC], & [DACB], \\ [CABD], & [ACDB], & [BDCA], & [DBAC]. \end{array}$$

The three sets of permutations just given correspond to the substitution groups of four letters denoted, respectively, by G_4, the "Vierergruppe"; G_8, a dihedron group; and G_{12}, the tetrahedron group.

In the symbolism of these groups we have:

$$
\begin{array}{lllll}
G_4: & 1, & (AB)(CD), & (AC)(BD), & (AD)(BC); \\
G_8: & 1, & (AB)(CD), & (AC)(BD), & (AD)(BC), \\
 & (AB), & (CD), & (ACBD), & (ADBC); \\
G_{12}: & 1, & (AB)(CD), & (AC)(BD), & (AD)(BC), \\
 & (ABC), & (ACD), & (BDC), & (ADB), \\
 & (ACB), & (BCD), & (ABD), & (ADC).
\end{array}
$$

If the substitutions of G_4 be denoted by ι_0, ι_1, ι_2, ι_3, respectively, then G_8 is obtained by adjoining to G_4 the transposition $\sigma = (AB)$, and G_{12} is obtained by adjoining to G_4 the circular substitution $\tau = (ABC)$. This composition of the groups will be useful in the following discussion:

$$G_4: \quad \iota_0, \quad \iota_1, \quad \iota_2, \quad \iota_3. \tag{1}$$

$$G_8: \quad \left.\begin{array}{llll} \iota_0, & \iota_1, & \iota_2, & \iota_3, \\ \sigma\iota_0, & \sigma\iota_1, & \sigma\iota_2, & \sigma\iota_3. \end{array}\right\} \tag{2}$$

$$G_{12}: \quad \left.\begin{array}{llll} \iota_0, & \iota_1, & \iota_2, & \iota_3, \\ \tau\iota_0, & \tau\iota_1, & \tau\iota_2, & \tau\iota_3, \\ \tau^2\iota_0, & \tau^2\iota_1, & \tau^2\iota_2, & \tau^2\iota_3. \end{array}\right\} \tag{3}$$

2. *The decomposition of the quartic into factors.* The transformations are considered here as those which carry a binary quartic into a multiple of itself. In a paper published in the AMERICAN JOURNAL, Vol. XVII, page 185, Professor STUDY has discussed fully the irrational covariants of the quartic, and has incidentally derived the expressions of the transformations of G_4. His further results enable us to write the transformations of G_8 and G_{12}. For the sake of clearness I quote as much of the invariant theory of the quartic and of Professor STUDY'S paper as is necessary for the purpose, adopting throughout the notation of the latter.

The rational covariant and invariant system of the quartic:

$$\left.\begin{array}{l}
f = (ax)^4 = (a'x)^4 = \ldots\ldots = (a_1 x_2 - a_2 x_1)^4, \\
h = (hx)^4 = (h'x)^4 = \ldots\ldots = \tfrac{1}{2}(aa')^2 (ax)^2 (a'x)^2, \\
t = (tx)^6 = (t'x)^6 = \ldots\ldots = 2(ah)(ax)^3(hx)^3, \\
g_2 = \tfrac{1}{2}(aa')^4, \qquad g_3 = \tfrac{1}{3}(ah)^4 = \tfrac{1}{6}(aa')^2(aa'')^2(a'a'')^2.
\end{array}\right\} \tag{4}$$

The discriminant takes the form

$$G = \frac{g_2^3 - 27\, g_3^2}{16}, \tag{5}$$

and the cubic resolvent,

$$4\, e^3 - g_2\, e - g_3 = 0, \tag{6}$$

with roots denoted by e_λ, e_μ, e_ν. The forms $h + e_\lambda f$, $h + e_\mu f$ and $h + e_\nu f$ are perfect squares, and they are factors of t^2. The quadratic factors of t are, however, given the following form for the sake of certain invariant properties:

$$\left.\begin{aligned} l &= (lx)^2 = \frac{1}{s_\lambda}\sqrt{-h - e_\lambda f}, \\ m &= (mx)^2 = \frac{1}{s_\mu}\sqrt{-h - e_\mu f}, \\ n &= (nx)^2 = \frac{1}{s_\nu}\sqrt{-h - e_\nu f}; \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} s_\lambda &= -\sqrt{e_\nu - e_\lambda}\,.\,\sqrt{e_\lambda - e_\mu}\,, \\ s_\mu &= -\sqrt{e_\lambda - e_\mu}\,.\,\sqrt{e_\mu - e_\nu}\,, \\ s_\nu &= -\sqrt{e_\mu - e_\nu}\,.\,\sqrt{e_\nu - e_\lambda}\,. \end{aligned}\right\} \tag{8}$$

Then is

$$\frac{t}{2} = -(e_\mu - e_\nu)(e_\nu - e_\lambda)(e_\lambda - e_\mu)\, lmn = -\sqrt{G}\,.\, lmn. \tag{9}$$

For the invariants and covariants of the quadratics l, m, n, we have

$$\left.\begin{aligned} (mn)_1 &= (mn)(mx)(nx) = -(lx)^2, \\ (nl)_1 &= (nl)(nx)(lx) \quad = -(mx)^2, \\ (lm)_1 &= (lm)(lx)(mx) \quad = -(nx)^2; \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} \tfrac{1}{2}(ll')^2 &= \tfrac{1}{2}(mm')^2 = \tfrac{1}{2}(nn')^2 = 1, \\ (mn)^2 &= (nl)^2 = (lm)^2 = 0. \end{aligned}\right\} \tag{11}$$

The decomposition of f into quadratic factors is derived from equations (7), and in fact in three forms corresponding to the three ways of pairing the linear factors:

$$\left.\begin{aligned} f &= -\frac{1}{e_\mu - e_\nu}(s_\mu m + s_\nu n)(s_\mu m - s_\nu n) \\ &= -\frac{1}{e_\nu - e_\lambda}(s_\nu n + s_\lambda l)(s_\nu n - s_\lambda l) \\ &= -\frac{1}{e_\lambda - e_\mu}(s_\lambda l + s_\mu m)(s_\lambda l - s_\mu m). \end{aligned}\right\} \tag{12}$$

The decomposition of f into linear factors is given as follows. We set

$$f = 4(r_0 x)(r_1 x)(r_2 x)(r_3 x), \tag{13}$$

and have for the squares of these linear factors (see the paper cited, p. 210):

$$\left.\begin{aligned} -2(r_0 x)^2 &= \sqrt{e_\mu - e_\nu}\,.\,l + \sqrt{e_\nu - e_\lambda}\,.\,m + \sqrt{e_\lambda - e_\mu}\,.\,n, \\ -2(r_\lambda x)^2 &= \sqrt{e_\mu - e_\nu}\,.\,l - \sqrt{e_\nu - e_\lambda}\,.\,m - \sqrt{e_\lambda - e_\mu}\,.\,n, \\ -2(r_\mu x)^2 &= -\sqrt{e_\mu - e_\nu}\,.\,l + \sqrt{e_\nu - e_\lambda}\,.\,m - \sqrt{e_\lambda - e_\mu}\,.\,n, \\ -2(r_\nu x)^2 &= -\sqrt{e_\mu - e_\nu}\,.\,l - \sqrt{e_\nu - e_\lambda}\,.\,m + \sqrt{e_\lambda - e_\mu}\,.\,n. \end{aligned}\right\} \tag{14}$$

The simultaneous invariants of these forms are given by the formulæ

$$\left.\begin{aligned} &(r_0 r_\lambda)(r_\mu r_\nu) = -(e_\mu - e_\nu); \quad (r_0 r_\lambda)^2 = (r_\mu r_\nu)^2 = e_\mu - e_\nu, \\ &(r_\mu r_\nu)(r_\nu r_\lambda)(r_\lambda r_\mu) = (r_\lambda r_0)(r_0 r_\mu)(r_\mu r_\lambda) = \sqrt[4]{G}, \\ &(r_\nu r_\lambda)(r_\lambda r_0)(r_0 r_\nu) = (r_0 r_\mu)(r_\mu r_\nu)(r_\nu r_0) = -\sqrt[4]{G}. \end{aligned}\right\} \tag{15}$$

In the formulæ (14) the signs of the radicals may be arbitrarily chosen, while that of $\sqrt[4]{G}$ is then determined by

$$\sqrt[4]{G} = \sqrt{e_\mu - e_\nu}\,.\,\sqrt{e_\nu - e_\lambda}\,.\,\sqrt{e_\lambda - e_\mu}\,.$$

3. *Transformations expressed in symbolic notation.* The binary bilinear form in cogredient variables, $(dx)(\delta y)$, set equal to zero, will, if the discriminant does not vanish, establish a projective correspondence between the points of the binary domain. To a point x (that is, $x_1 : x_2$), there corresponds a point x', the vanishing point of $(dx)(\delta y) = (x'y) = 0$. The linear invariant of the form is $i = (d\delta)$, and the discriminant, $j = \frac{1}{2}(dd')(\delta\delta')$. The identity transformation is given by $(xy) = 0$. If $s_1 = (d_1 x)(\delta_1 y) = 0$ and $s_2 = (d_2 x)(\delta_2 y) = 0$ are two non-degenerate transformations, then their product is given by

$$s_1 s_2 = (d_1 x)(d_2 \delta_1)(\delta_2 y) = 0.$$

The condition that a transformation is involutory or of period 2, is $i = 0$; of period 3, $i^2 - j = 0$; of period 4, $i^2 - 2j = 0$; and so on.

4. *The transformations of the general case* are, as stated above, derived in the paper from which I have quoted. It is a well-known property of the covariant t of the quartic that the vanishing points of each of the quadratic factors are the double points of an involution, of which every form of the pencil $\varkappa f + \lambda h$ furnishes two pairs of points. In particular, to each of the three ways of pairing the points of f correspond in this manner, as double points of an involution, the root point of a quadratic factor of t.

The polar form $(\alpha x)(\alpha y)$, set equal to zero, is, if the discriminant $\frac{1}{2}(\alpha\alpha')^2$ does not vanish, a transformation pairing the points of the line according to an involution whose double points are given by the equation $(\alpha x)^2 = 0$. We have in this way for the bilinear forms representing the transformations of the group leaving the general quartic unaltered:

$$G_4: \quad \iota_0 = (xy), \quad \iota_1 = (lx)(ly), \quad \iota_2 = (mx)(my), \quad \iota_3 = (nx)(ny). \tag{16}$$

5. *The transformations for the harmonic case* may be derived from the consideration of the quadratic factors of f, (12). If the points of f form a harmonic quadruple, the invariant g_3 vanishes, and conversely. The cubic resolvent (6) then becomes

$$4e^3 - g_2 e = 0,$$

with roots 0, $\frac{1}{2}\sqrt{g_2}$ and $-\frac{1}{2}\sqrt{g_2}$. According to the naming of the roots, we have different forms of the transformations. Setting

$$e_\lambda = 0, \quad e_\mu = \tfrac{1}{2}\sqrt{g_2}, \quad e_\nu = -\tfrac{1}{2}\sqrt{g_2},$$

the decomposition (12) becomes

$$\left.\begin{aligned} f &= \tfrac{1}{2}\sqrt{g_2}(l+m)(l-m) \\ &= -\sqrt{g_2}\left(n + \frac{1}{\sqrt{-2}}l\right)\left(n - \frac{1}{\sqrt{-2}}l\right) \\ &= \tfrac{1}{2}\sqrt{g_2}(l + \sqrt{-2}\,m)(l - \sqrt{-2}\,m); \end{aligned}\right\} \tag{17}$$

where now

$$\left.\begin{aligned} l &= -\frac{2}{\sqrt{g_2}}\sqrt{-h}, \\ m &= -\sqrt{\frac{2}{g_2}}\sqrt{-h - \tfrac{1}{2}\sqrt{g_2}\cdot f}, \\ n &= -\sqrt{\frac{2}{g_2}}\sqrt{-h + \tfrac{1}{2}\sqrt{g_2}\cdot f}. \end{aligned}\right\} \tag{18}$$

The second and third decompositions of f in (17) represent a pairing of the linear factors corresponding to the double ratio $\frac{1}{2}$ or 2. In the first this double ratio is -1, and from this we derive the desired transformations. The involutory transformation, namely, whose double points are given by $m + n = 0$, as well as that whose double points are given by $m - n = 0$, will interchange one pair

of the harmonic points, leaving the other points unaltered. For the group G_8 we have then, by (2), only to adjoin to G_4 the transformation

$$\sigma = (mx)(my) + (nx)(ny).$$

$$G_8: \quad \left.\begin{array}{l} \iota_0 = (xy), \quad \iota_1 = (lx)(ly), \quad \iota_2 = (mx)(my), \quad \iota_3 = (nx)(ny), \\ \sigma\iota_0 = (mx)(my) + (nx)(ny), \quad \sigma\iota_1 = (mx)(my) - (nx)(ny), \\ -\sigma\iota_2 = (lx)(ly) + (xy), \quad -\sigma\iota_3 = -(lx)(ly) + (xy). \end{array}\right\} \quad (19)$$

The transformations $\sigma\iota_0$ and $\sigma\iota_1$ are involutory, the linear invariant i vanishing by virtue of (11); $\sigma\iota_2$ and $\sigma\iota_3$ are of period 4, as for each $i = -2$ and $j = 2$, and hence $\iota^2 - 2j = 0$.

5. *The transformations of the equianharmonic case* are derived from the decomposition of f into linear factors. We know that for this case there exist transformations which permute cyclically any three of the points, leaving the fourth unaltered.

We first derive a general formula for the transformation which takes the points x_1, x_2, x_3, into y_1, y_2, y_3.* This is given by the elimination of the coefficients $d_\alpha \delta_\beta$ from the equations

$$(dx)(\delta y) = 0, \quad (dx_1)(\delta y_1) = 0, \quad (dx_2)(\delta y_2) = 0, \quad (dx_3)(\delta y_3) = 0;$$

which gives as a result the transformation

$$(dx)(\delta y) = (x_1 x_3)(y_1 y_2)(x_2 x)(y_3 y) - (x_1 x_2)(y_1 y_3)(x_3 x)(y_2 y) = 0. \quad (20)$$

This is a statement of the equality of the double ratios $[x_1 x_2 x_3 x]$ and $[y_1 y_2 y_3 y]$, and as such may be given other equivalent forms. Setting now $y_1 = x_2$, $y_2 = x_3$, $y_3 = x_1$, we have for the *cyclic* transformation,

$$(dx)(\delta y) = (x_1 x_2)^2 (x_3 x)(x_3 y) - (x_3 x_1)(x_2 x_3)(x_2 x)(x_1 y).$$

This may be given a symmetric form. Permuting cyclically the letters x_1, x_2, x_3 and adding the results, we obtain

$$\begin{aligned} 3(dx)(\delta y) = {} & (x_2 x_3)^2 (x_1 x)(x_1 y) + (x_3 x_1)^2 (x_2 x)(x_2 y) + (x_1 x_2)^2 (x_3 x)(x_3 y) \\ & - (x_3 x_1)(x_1 x_2)(x_3 x)(x_2 y) - (x_1 x_2)(x_2 x_3)(x_1 x)(x_3 y) \\ & - (x_2 x_3)(x_3 x_1)(x_2 x)(x_1 y). \end{aligned}$$

The last three terms may be easily reduced in pairs by use of the fundamental identity of binary forms. Thus, for example,

$$\begin{aligned} & -(x_1 x_2)(x_2 x_3)(x_1 x)(x_3 y) - (x_2 x_3)(x_3 x_1)(x_2 x)(x_1 y) \\ & \qquad = (x_1 x_2)^2 (x_3 x)(x_3 y) + (x_2 x_3)(x_3 x_1)(x_1 x_2) \,.\, (xy). \end{aligned}$$

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* By $x_1, x_2, x_3, y_1, y_2, y_3$ are meant $x_{11} : x_{12}$, $x_{21} : x_{22}$, $x_{31} : x_{32}$, $y_{11} : y_{12}$, $y_{21} : y_{22}$, $y_{31} : y_{32}$.

Substituting these results, we have finally

$$\left.\begin{aligned}2(dx)(\delta y) = (x_2 x_3)^2 (x_1 x)(x_1 y) + (x_3 x_1)^2 (x_2 x)(x_2 y) \\ + (x_1 x_2)^2 (x_3 x)(x_3 y) + (x_2 x_3)(x_3 x_1)(x_1 x_2) \,.\, (xy).\end{aligned}\right\} \quad (21)$$

This is the general form of a transformation determined by the passing of x_1, x_2, x_3 into x_2, x_3, x_1, respectively.

Writing in this formula r_λ, r_μ, r_ν [see (13)] for x_1, x_2, x_3, we have the transformation sought:

$$\left.\begin{aligned}2(dx)(\delta y) = (r_\mu r_\nu)^2 (r_\lambda x)(r_\lambda y) + (r_\nu r_\lambda)^2 (r_\mu x)(r_\mu y) \\ + (r_\lambda r_\mu)^2 (r_\nu x)(r_\nu y) + (r_\mu r_\nu)(r_\nu r_\lambda)(r_\lambda r_\mu) \,.\, (xy).^*\end{aligned}\right\} \quad (22)$$

By use of (14) and (15) this may be written in terms of l, m, n, e_λ, e_μ, e_ν. From these formulæ we have

$$\begin{aligned}&-2(r_0 r_\lambda)^2 (r_\lambda x)(r_\lambda y) = \\ &\quad (e_\mu - e_\nu)\left[\sqrt{e_\mu - e_\nu} \,.\, (lx)(ly) - \sqrt{e_\nu - e_\lambda} \,.\, (mx)(my) - \sqrt{e_\lambda - e_\mu} \,.\, (nx)(ny)\right], \\ &-2(r_0 r_\mu)^2 (r_\mu x)(r_\mu y) = \\ &\quad (e_\nu - e_\lambda)\left[-\sqrt{e_\mu - e_\nu} \,.\, (lx)(ly) + \sqrt{e_\nu - e_\lambda} \,.\, (mx)(my) - \sqrt{e_\lambda - e_\mu} \,.\, (nx)(ny)\right], \\ &-2(r_0 r_\nu)^2 (r_\nu x)(r_\nu y) = \\ &\quad (e_\lambda - e_\mu)\left[-\sqrt{e_\mu - e_\nu} \,.\, (lx)(ly) - \sqrt{e_\nu - e_\lambda} \,.\, (mx)(my) + \sqrt{e_\lambda - e_\mu} \,.\, (nx)(ny)\right], \\ &-2(r_\mu r_\nu)(r_\nu r_\lambda)(r_\lambda r_\mu) \,.\, (xy) = -2\sqrt{e_\mu - e_\nu} \,.\, \sqrt{e_\nu - e_\lambda} \,.\, \sqrt{e_\lambda - e_\mu} \,.\, (xy);\end{aligned}$$

whence, by an easy reduction, (22) becomes

$$\left.\begin{aligned}-2(dx)(\delta y) = (\sqrt{e_\mu - e_\nu})^3 \,.\, (lx)(ly) + (\sqrt{e_\nu - e_\lambda})^3 \,.\, (mx)(my) \\ + (\sqrt{e_\lambda - e_\mu})^3 \,.\, (nx)(ny) - \sqrt{e_\mu - e_\nu} \,.\, \sqrt{e_\nu - e_\lambda} \,.\, \sqrt{e_\lambda - e_\mu} \,.\, (xy).\end{aligned}\right\} \quad (23)$$

The equianharmonic case is characterized by the vanishing of the invariant g_2. The cubic resolvent becomes then

$$4e^3 - g_3 = 0,$$

and its roots,

$$e_\lambda = \varepsilon_1 \sqrt[3]{\frac{g_3}{4}}, \quad e_\mu = \varepsilon_2 \sqrt[3]{\frac{g_3}{4}}, \quad e_\nu = \varepsilon_3 \sqrt[3]{\frac{g_3}{4}}, \quad (24)$$

* This transformation will hold r_0 unaltered if $(r_0 r_\lambda)^4 + (r_0 r_\mu)^4 + (r_0 r_\nu)^4 = 0$; or by (14), if $(e_\mu - e_\nu)^2 + (e_\nu - e_\lambda)^2 + (e_\lambda - e_\mu)^2 = 0$. In the equianharmonic case [see (24)], this condition becomes $(\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2 + (\varepsilon_1 - \varepsilon_2)^2 = 0$, and this is a numerical identity.

where ε_1, ε_2, ε_3 are the cube roots of unity, named arbitrarily. The radical expressions in (23),

$$(\sqrt{e_\mu - e_\nu})^3, \quad (\sqrt{e_\nu - e_\lambda})^3, \quad (\sqrt{e_\lambda - e_\mu})^3, \quad \sqrt{e_\mu - e_\nu} \cdot \sqrt{e_\nu - e_\lambda} \cdot \sqrt{e_\lambda - e_\mu}, \tag{25}$$

take now, except for sign, the same value, and indeed each is equal to $\pm \frac{1}{2} \sqrt{(\sqrt{-3})^3 g_3}$, or each is equal to $\pm \frac{1}{2} \sqrt{-(\sqrt{-3})^3 g_3}$. From the transformation (23) we may omit then a common factor, the common value of the expressions (25). We shall have different transformations according to the choice of signs of the radicals in (14), which was arbitrary; there will in fact be exactly eight which are essentially different, each of which will transform the quartic into a multiple of itself. These may also be derived by (3); we have in fact, putting

$$\tau = -(lx)(ly) - (mx)(my) - (nx)(ny) + (xy),$$

$$G_{12}: \left.\begin{aligned} \iota_0 = (xy), \quad \iota_1 = (lx)(ly), \quad \iota_2 &= (mx)(my), \quad \iota_3 = (nx)(ny), \\ \tau\iota_0 &= -(lx)(ly) - (mx)(my) - (nx)(ny) + (xy), \\ \tau\iota_1 &= (lx)(ly) - (mx)(my) + (nx)(ny) + (xy), \\ \tau\iota_2 &= (lx)(ly) + (mx)(my) - (nx)(ny) + (xy), \\ \tau\iota_3 &= -(lx)(ly) + (mx)(my) + (nx)(ny) + (xy), \\ -\tfrac{1}{2}\tau^2\iota_0 &= (lx)(ly) + (mx)(my) + (nx)(ny) + (xy), \\ -\tfrac{1}{2}\tau^2\iota_1 &= -(lx)(ly) - (mx)(my) + (nx)(ny) + (xy), \\ -\tfrac{1}{2}\tau^2\iota_2 &= (lx)(ly) - (mx)(my) - (nx)(ny) + (xy), \\ -\tfrac{1}{2}\tau^2\iota_3 &= -(lx)(ly) + (mx)(my) - (nx)(ny) + (xy). \end{aligned}\right\} \tag{26}$$

The respective invariants of the last eight of these transformations have the same value. Writing

$$(dx)(\delta y) = \pm (lx)(ly) \pm (mx)(my) \pm (nx)(ny) + (xy),$$

we have

$$\begin{aligned} i &= (d\delta) = 2, \\ j &= \tfrac{1}{2}(dd')(\delta\delta') \\ &= \tfrac{1}{2}\{\pm (dl)(\delta l) \pm (dm)(\delta m) \pm (dn)(\delta n) + (d\delta)\} \\ &= \tfrac{1}{2}\{(ll')^2 + (mm')^2 + (nn')^2 + 2\} = 4. \end{aligned}$$

Hence, in every instance $i^2 - j = 0$; that is, the transformations $\tau\iota_\kappa$ are all of period 3.

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